FIXED POINT THEOREMS FOR NON-LIPSCHITZIAN MAPPINGS OF ASYMPTOTICALLY NONEXPANSIVE TYPE†

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ABSTRACT

Let X be a Banach space, K a nonempty, bounded, closed and convex subset of X, and suppose $T : K \to K$ satisfies:

(*) for each $x \in K$, $\limsup_{i\to\infty} \{\sup_{y\in K}[\,\|T^i x - T^i y\| - \|x-y\|\,]\} \leqq 0$.

If T^N is continuous for some positive integer N, and if either (a) X is uniformly convex, or (b) K is compact, then T has a fixed point in K. The former generalizes a theorem of Goebel and Kirk for asymptotically nonexpansive mappings. These are mappings $T: K \to K$ satisfying, for i sufficiently large, $\|T^i x - T^i y\| \leqq k_i \|x - y\|$, $x, y \in K$, where $k_i \to 1$ as $i \to \infty$. The precise assumption in (a) is somewhat weaker than uniform convexity, requiring only that Goebel's characteristic of convexity, $\varepsilon_0(X)$, be less than one.

Let X be a Banach space, $K \subseteq X$. A mapping $T: K \to K$ is called *asymptotically nonexpansive* on K [5] if there exists a sequence $\{k_i\}$ of constants such that $k_i \to 1$ as $i \to \infty$ and for which

$$\| T^i x - T^i y \| \leqq k_i \| x - y \|, \qquad x, y \in K, \; i \geqq N_0.$$

It was proved in [5] that if X is uniformly convex and if K is bounded, closed, and convex, then such a mapping must have a fixed point. This is, of course, a natural generalization of the fixed point theorem of Browder-Göhde-Kirk [1], [8], [11] for nonexpansive mapping.

Our purpose in this paper is twofold. First we substantially weaken the assumption of asymptotic nonexpansiveness of T by replacing it with an as-

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sumption, (2) below, which may hold even if none of the iterates of T is Lipschitzian. Although we assume that at least one of its iterates is continuous, the mapping itself need not be. In addition, we obtain one of our results in a class of spaces which properly includes the uniformly convex spaces.

Our second objective is to obtain an analogous result for compact convex K with no underlying assumptions on the norm of the space. Again the assumption is that $T: K \to K$ satisfy (2) and T^N be continuous for some N. This theorem provides a new result even for asymptotically nonexpansive mappings, and because T is not assumed continuous it does not follow directly from the Schauder theorem.

A generalization of the result of [5] which retains the feature that iterates of T are Lipschitzian but only requires that these Lipschitz constants be sufficiently near one (while perhaps being bounded away from one) is given in [6]. It is assumed that X is uniformly convex in [6], but this result itself has subsequently been generalized in [7] to the wider class of spaces considered below.

The *modulus of convexity* of X is the function $\delta: [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf[1 - \tfrac{1}{2}\| x + y \| : x, y \in X, \ \| x \|, \ \| y \| \leqq 1, \ \| x - y \| \geqq \varepsilon].$$

Let

$$\varepsilon_0(X) = \sup\{\varepsilon : \delta(\varepsilon) = 0\}.$$

The number $\varepsilon_0(X)$ is called the *characteristic of convexity* of X [4]. In Theorem 1 we assume X satisfies $\varepsilon_0(X) < 1$. It is known (see Goebel [4]) that this implies X is uniformly non-square, hence reflexive [10]. Also, X is uniformly convex [2] if $\delta(\varepsilon) > 0$ whenever $\varepsilon > 0$; hence $\varepsilon_0(X) = 0$ for such spaces and so Theorem 1 holds for X uniformly convex.

It is known (see [9], [12]) that the modulus of convexity is continuous and increasing on $[\varepsilon_0, 2)$ and moreover [13], [14], the inequalities

$$\| x \| \leqq d, \ \| y \| \leqq d, \ \| x - y \| \geqq \varepsilon$$

imply

(1) $$\tfrac{1}{2}\| x + y \| \leqq (1 - \delta(\varepsilon / d))d.$$

For $x \in X$, $S(x; r)$ will denote the closed spherical ball $\{y \in X : \| x - y \| \leqq r\}$.

In each of our theorems we assume $T: K \to K$ satisfies:

(2) for each $x \in K$, $\limsup_{i \to \infty} \{ \sup_{y \in K} [\| T^i x - T^i y \| - \| x - y \|]\} \leqq 0.$

We compare this assumption with asymptotic nonexpansiveness in the remark following the proof of Theorem 2.

THEOREM 1. *Let X be a Banach space for which $\varepsilon_0 = \varepsilon_0(X) < 1$ and let $K \subseteq X$ be nonempty, bounded, closed and convex. Suppose $T: K \to K$ has the property that T^N is continuous for some positive integer N, and suppose T satisfies* (2). *Then T has a fixed point in K.*

THEOREM 2. *Let K be a nonempty, compact and convex subset of the Banach space X. Suppose $T: K \to K$ has the property that T^N is continuous for some positive integer N, and suppose T satisfies* (2). *Then T has a fixed point in K.*

The proof of Theorem 1 follows closely that of Goebel and Kirk in [6], hinging on properties of the modulus of convexity of X, while Theorem 2 requires a more topological argument.

PROOF OF THEOREM 1. Let $x \in K$ be fixed. As seen in [5], [6] there exists a number $\rho_0 = \rho_0(x) \geqq 0$ which is minimal with respect to the property: for each $\varepsilon > 0$ there exists an integer k such that

$$K \cap \left(\bigcap_{i=k}^{\infty} S(T^i x; \rho_0 + \varepsilon) \right) \neq \emptyset.$$

Letting

$$C_\varepsilon = \bigcup_{k=1}^{\infty} \left(\bigcap_{i=k}^{\infty} S(T^i x; \rho_0 + \varepsilon) \right),$$

then for $\varepsilon > 0$ the set C_ε is nonempty, bounded and convex; hence by reflexivity of X the closure $\bar{C}_\varepsilon$ of C_ε is weakly compact and

$$C = \bigcap_{\varepsilon > 0} (\bar{C}_\varepsilon \cap K) \neq \emptyset.$$

Now let $z \in C$, and let

$$d(z) = \limsup_{i \to \infty} \| z - T^i z \|.$$

Suppose $\rho_0(x) = 0$. Then clearly $T^n x \to z$ as $n \to \infty$. Let $\eta > 0$ and using (2) choose M so that $i \geqq M$ implies

$$\sup_{y \in K} [\| T^i z - T^i y \| - \| z - y \|] \leqq \tfrac{1}{3}\eta.$$

Given $i \geqq M$, since $T^n x \to z$ there exists $m > i$ such that $\| T^m x - z \| \leqq \frac{1}{3}\eta$ and $\| T^{m-i} x - z \| \leqq \frac{1}{3}\eta$. Thus if $i \geqq M$,

$$\begin{aligned}\| z - T^i z \| &\leqq \| z - T^m x\| + \| T^m x - T^i z \| \\ &\leqq \| z - T^m x \| + \| T^i z - T^i(T^{m-i}x) \| - \| z - T^{m-i}x \| \\ &\quad + \| z - T^{m-i}x \| \\ &\leqq \tfrac{1}{3}\eta + \sup_{y \in K} [\| T^i z - T^i y \| - \| z - y \|] + \tfrac{1}{3}\eta \\ &\leqq \eta.\end{aligned}$$

This proves $T^n z \to z$ as $n \to \infty$, that is, $d(z) = 0$. But $d(z) = 0$ implies $T^{Ni}z \to z$ as $i \to \infty$ and with continuity of T^N this yields $T^N z = z$. Thus

(3) $$Tz = T(T^{Ni}z) = T^{Ni+1}z \to z \text{ as } i \to \infty,$$

and $Tz = z$. Therefore we may assume $\rho_0(x) > 0$ and $d(z) > 0$. (In fact, we may assume this *for any* $x, z \in K$.)

Now let $\varepsilon > 0$, $\varepsilon \leqq d(z)$. By the definition of ρ_0 there exists an integer N^* such that if $i \geqq N^*$ then

$$\| z - T^i x \| \leqq \rho_0 + \varepsilon,$$

and by (2) there exists N^{**} such that if $i \geqq N^{**}$ then

$$\sup_{y \in K} [\| T^i z - T^i y \| - \| z - y \|] \leqq \varepsilon.$$

Select j so that $j \geqq N^{**}$ and so that

$$\| z - T^j z \| \geqq d(z) - \varepsilon.$$

Thus if $i - j \geqq N^*$,

$$\begin{aligned}\| T^j z - T^i x \| &= \{\| T^j z - T^j(T^{i-j}x) \| - \| z - T^{i-j}x \|\} + \| z - T^{i-j}x \| \\ &\leqq \varepsilon + (\rho_0 + \varepsilon) \\ &= \rho_0 + 2\varepsilon.\end{aligned}$$

Letting $m = \frac{1}{2}(z + T^j z)$ we have by property (1) of the modulus of convexity,

$$\| m - T^i x \| \leqq \left(1 - \delta\left(\frac{d(z) - \varepsilon}{\rho_0 + 2\varepsilon}\right)\right)(\rho_0 + 2\varepsilon), \qquad i \geqq N^* + j.$$

By the minimality of ρ_0 this implies

$$\rho_0 \leqq \left(1 - \delta\left(\frac{d(z) - \varepsilon}{\rho_0 + 2\varepsilon}\right)\right)(\rho_0 + 2\varepsilon);$$

letting $\varepsilon \to 0$,

$$\rho_0 \leqq \left(1 - \delta\left(\frac{d(z)}{\rho_0}\right)\right)\rho_0.$$

This implies $1 - \delta(d(z)/\rho_0) \geqq 1$ and hence $\delta(d(z)/\rho_0) = 0$. It follows from the definition of ε_0 that $d(z)/\rho_0 \leqq \varepsilon_0$. Hence

$$d(z) \leqq \varepsilon_0 \rho_0(x)$$

and letting $d(x) = \limsup_{i\to\infty} \| x - T^i x \|$ we have $\rho_0(x) \leqq d(x)$ so

$$d(z) \leqq \varepsilon_0 d(x). \tag{4}$$

Also notice that $\| z - x \| \leqq d(x) + \rho_0(x) \leqq 2d(x)$.

The proof is completed almost precisely as in [6]. We include the details for completeness.

Fix $x_0 \in K$ and define the sequence $\{x_n\}$ by $x_{n+1} = z(x_n)$, $n = 0, 1, 2, \cdots$, where $z(x_n)$ is obtained from x_n in the same manner as $z(x)$ from x. If for any n we have $\rho(x_n) = 0$ then, as seen above, $Tx_{n+1} = x_{n+1}$. Otherwise we have by (4)

$$\| x_{n+1} - x_n \| \leqq 2d(x_n) \leqq 2(\varepsilon_0)^n d(x_0)$$

and since $\varepsilon_0 < 1$, $\{x_n\}$ is a Cauchy sequence. Therefore there exists $y \in K$ such that $x_n \to y$ as $n \to \infty$. Also

$$\begin{aligned} \| y - T^i y \| &\leqq \| y - x_n \| + \| x_n - T^i x_n \| + \| T^i x_n - T^i y \| \\ &\leqq \| y - x_n \| + \| x_n - T^i x_n \| + [\| T^i x_n - T^i y \| - \| x_n - y \|] + \| x_n - y \|. \end{aligned}$$

Thus

$$\begin{aligned} d(y) &= \limsup_{i\to\infty} \| y - T^i y \| \\ &\leqq \limsup_{i\to\infty} 2\| x_n - y \| + \limsup_{i\to\infty} \| x_n - T^i x_n \| \\ &\quad + \limsup_{i\to\infty} [\| T^i x_n - T^i y \| - \| x_n - y \|] \\ &\leqq d(x_n) + 2\| x_n - y \|. \end{aligned}$$

Since $x_n \to y$ and $d(x_n) \to 0$ as $n \to \infty$, this implies $d(y) = 0$. But as seen before (3) this implies $Ty = y$.

Proof of Theorem 2. Use Zorn's lemma to obtain a subset H of K which is minimal with respect to being nonempty, closed, convex, and satisfying:

(5) If $x \in H$ and w is a subsequential limit of $\{T^n x\}$, then $w \in H$.

Now let $H_N \subseteq H$ be minimal with respect to being nonempty, closed, convex, and satisfying:

(6) If $x \in H_N$ and w is a subsequential limit of $\{T^{N_i}x\}_{i=1}^{\infty}$, then $w \in H_N$.

If $\operatorname{diam}(H_N) = 0$ then clearly H_N consists of a single point which is fixed under the mapping $F = T^N$. To see that this must be the case, suppose $\operatorname{diam}(H_N) > 0$. Note that if

$$S = \{z \in H_N: z \text{ is a subsequential limit of } \{F^n x\} \text{ for some fixed } x \in H_N\}$$

then $F: S \to S$. Moreover, since F is continuous, S is closed. Thus we may select a minimal, nonempty, closed subset S_N of H_N which is invariant under F, and because F is continuous, F maps S_N *onto* S_N. As before, if $\operatorname{diam}(S_N) = 0$ then S_N consists of a single point which is fixed under F. On the other hand, if $\delta_1 = \operatorname{diam}(S_N) > 0$ then, as shown by De Marr [3], there exists a number $r_1 < \delta_1$ such that for some $x \in H_N$, $\sup\{\|x - z\|: z \in S_N\} \leqq r_1$. Let

$$C_N = \{x \in H_N: S_N \subseteq B(x; r_1)\}$$

where $B(x; r_1)$ denotes the closed ball centered at x with radius r_1. Clearly C_N is nonempty, closed, and convex and moreover, because $\operatorname{diam}(S_N) = \delta_1 > r_1$ and C_N cannot contain points of S_N whose distance exceeds r_1, it follows that C_N is a *proper* subset of H_N.

ASSERTION. C_N satisfies (6). To see this, let $z \in C_N$ and suppose $\lim_{i\to\infty} F^{n_i}z = w$. We must show $w \in C_N$. If $y \in S_N$ then, since F maps S_N onto S_N, for each i there exists $u_{n_i} \in S_N$ such that $y = F^{n_i}u_{n_i}$. Thus

$$\|w - y\| \leqq \|w - F^{n_i}z\| + \|F^{n_i}z - F^{n_i}u_{n_i}\|.$$

Using (2),

$$\begin{aligned}
&\limsup_{i\to\infty} \{\|w - F^{n_i}z\| + \|F^{n_i}z - F^{n_i}u_{n_i}\|\} \\
&\qquad \leqq \limsup_{i\to\infty} \{\|F^{n_i}z - F^{n_i}u_{n_i}\| - \|z - u_{n_i}\|\} + \limsup_{i\to\infty} \|z - u_{n_i}\| \\
&\qquad \leqq r_1
\end{aligned}$$

and this implies $\|w - y\| \leqq r_1$. Since $w \in H_N$ by (6), this proves that $w \in C_N$.

The assertion along with the other properties of C_N shows that the minimality of H_N is contradicted if $\delta_1 > 0$; thus it must be the case that $\delta_1 = \operatorname{diam}(H_N) = 0$. Therefore we have established the existence of a point $x_0 \in H$ such that $T^N x_0 = x_0$.

To complete the proof, suppose $Tx_0 \neq x_0$ and let $S = \{x_0, Tx_0, \cdots, T^{N-1}x_0\}$. Then if $\delta_2 = \operatorname{diam}(S)$, as before there exists $r_2 < \delta_2$ such that for some $x \in H$, $\sup\{\|x - z\|: z \in S\} \leqq r_2$. Let

$$C = \{x \in H : S \subseteq B(x; r_2)\}.$$

Then C is nonempty, closed, convex, and moreover C is a proper subset of H. By following precisely the argument of the assertion (replacing F with T and removing the subscripts N) one sees that C satisfies (5), contradicting the minimality of H. Therefore $Tx_0 = x_0$.

REMARK. If K is bounded and if $T: K \to K$ is asymptotically nonexpansive in the sense of [5] then T satisfies (2).

PROOF. If T is asymptotically nonexpansive in the sense of [5] then there exists a sequence $\{k_i\}$ of constants such that $k_i \to 1$ as $i \to \infty$ and for which

$$\| T^i x - T^i y \| \leqq k_i \| x - y \|, \qquad x, y \in K, \ i \geqq N_0.$$

Thus

$$\| T^i x - T^i y \| - \| x - y \| \leqq (k_i - 1) \| x - y \| \leqq | k_i - 1 | \delta(K)$$

and

$$\limsup_{i \to \infty} \left\{ \sup_{y \in K} [\| T^i x - T^i y \| - \| x - y \|] \right\} \leqq \lim_{i \to \infty} | k_i - 1 | \delta(K) = 0.$$

Theorem 2 thus has the following corollary.

COROLLARY. *Suppose K is compact and convex and suppose $T: K \to K$ satisfies for $i \geqq N_0$,*

$$\| T^i x - T^i y \| \leqq k_i \| x - y \|, \qquad x, y \in K,$$

where $k_i \to 1$ as $i \to \infty$. Then T has a fixed point in K.

The converse of the remark is not true. Simple examples of real-valued functions f on the unit interval can be constructed which satisfy (2) (that is, such that $f^n(x) \to 0$ uniformly as $n \to \infty$) but for which f^i is not Lipschitzian for any integer i.

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